J. Fluid Mech. (1970), vol. 40, part 1, pp. 21–38
Printed in Great Britain

Magnetohydrodynamic free convection in a strong cross field

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(Received 13 January 1969 and in revised form 4 August 1969)

The problem of magnetohydrodynamic free convection of an electrically conducting fluid in a strong cross field is investigated. It is solved by using a singular perturbation technique. The solutions presented cover the range of Prandtl numbers from zero to order one. This includes both the important cases of liquid metals and ionized gases. A general examination is given of the role of the important parameters: Hartmann, Grashof and Prandtl numbers of the problem. This provides clear insight into its singular character and yields the correct expansion parameters. The boundary-layer approximations are derived from the complete Navier–Stokes and energy equations. The conditions for these approximations to be valid will be explicitly stated. Attention is given to 'power law' wall-temperatures and magnetic fields, and an assessment is given of the range of application.

1. Introduction

Several aspects of steady free convection of an electrically conducting fluid in magnetic fields have been discussed in recent years by a number of authors (Singh & Cowling 1963, Riley 1964, D'sa 1967). Riley was the first to recognize that the solution of the equations of magnetohydrodynamic free convection of a viscous flow upwards along a heated vertical semi-infinite plate, in the presence of a strong magnetic field acting in a normal direction to the boundary, requires the application of the method of matched asymptotic expansion. Riley's reasoning, which was given for a constant magnetic field and a constant wall-temperature, is essentially the following.

Near the leading edge the velocity is very small so that the magnetic force, which is proportional to the magnitude of the longitudinal velocity and acts in the opposite direction, is also very small. The fluid will be accelerated by buoyancy forces alone and balanced essentially by viscous shear forces. Now, it is known that in free convection fields in which no magnetic forces are present the longitudinal velocity increases as the root of the distance from the leading edge. This relationship will not hold if an appreciable magnetic field normal to the plate is present: the force induced by this field will retard the flow. Since the buoyancy force is constant, a state of balance will be attained when the magnetic force is also constant, i.e. when the longitudinal velocity is constant. Thus, *all* the fluid is entrained upwards and gradually approaches this constant characteristic velocity.

Viewing the plate from a distance it seems as if the plate were moving upwards with this constant velocity. Singh & Cowling and Riley showed, indeed, that in strong magnetic cross-fields the major (outer) part of the flow is described by Blasius's equation with boundary conditions for a moving plate in a fluid at rest. In the present problem the plate is not moving, so that near the boundary there has to be a thin (inner) layer in which the longitudinal velocity decreases to zero. In this layer the viscous stresses are of the same order of magnitude as the buoyancy and the magnetic forces.

In the present paper solutions to boundary-layer equations only will be given. The validity of the boundary-layer approximation will be carefully investigated by starting the analysis with the full Navier–Stokes and energy equations. The full range of Prandtl numbers from zero to unity will be covered. That includes the important case of magnetohydrodynamic free convection in liquid metals.

The solutions of Riley and D'sa were given for $P \sim 1$. As this problem involves three characteristic numbers, viz. the Hartmann number, the Grashof number and the Prandtl number, it was deemed necessary to give a general examination of the importance of each of these parameters. This investigation reveals that the solution requires the introduction of a double series expansion which is partly singular and partly regular. The theory will be applied to 'power law' type of wall-temperature variations and magnetic fields. An estimate will be made of the limitations of application. As in Riley's analysis, the Joule heating is not taken into consideration.

2. Inner and outer equations

As indicated in § 1, we shall solve the present problem in the region where the buoyancy and the magnetic forces balance each other. Taking into consideration that the order of magnitude of these forces is $g\beta(T_W - T_\infty)$ and AU respectively, where g is the acceleration due to gravity, β the coefficient of thermal expansion, T_W the wall temperature, T_∞ the ambient temperature, U the characteristic longitudinal velocity and A a factor of proportionality relating the magnetic force to the longitudinal velocity $\tilde{u}$, we find for the characteristic velocity

$$U = \frac{g\beta(T_W - T_\infty)}{A}. \tag{1}$$

For a systematic derivation of the magnetic force the reader is referred to Singh & Cowling (1963). Equation (1) will be seen to be the maximum velocity existing in the system.

Next, we shall try to derive boundary-layer equations for the flow. In order to be able to state explicitly that the conditions for these approximations are valid, it is necessary to consider initially the full Navier–Stokes and energy equations. It will be also assumed that the Boussinesq approximations are applicable throughout, i.e. the density variations are taken into account only insofar as they affect the buoyancy term. This approach is valid if

$$\frac{T_W - T_\infty}{T_\infty} \ll 1. \tag{2}$$

The original equations are rendered dimensionless through the introduction of the following set of equations, where dimensional variables are denoted by a tilde,

$$\left.\begin{aligned} \tilde{x} &= lx, \quad \tilde{y} = ly, \quad \tilde{u} = Uu, \quad \tilde{v} = Uv, \\ \tilde{T} &= T_\infty + (T_W - T_\infty)\, T, \quad \tilde{p} = p_\infty + \rho_\infty U^2 p. \end{aligned}\right\} \tag{3}$$

Here $\tilde{x}$ measures distance upwards along the vertical plate ($\tilde{x} = 0$ at the leading edge), $\tilde{y}$ measures distance normal to the plate ($\tilde{y} = 0$ at the plate), $\tilde{u}$ and $\tilde{v}$ are the velocities in the $\tilde{x}$ and $\tilde{y}$ direction respectively, $\tilde{T}$ is the temperature, $\tilde{\rho}$ the density and $\tilde{p}$ the pressure, l is a characteristic length chosen to render x of order unity, ∞ refers to ambient conditions, p_∞ is the hydrostatic pressure $(p_{\text{ref}} - \rho_\infty g\tilde{x})$ and U is taken from (1). It may be noted that this process can only be used for qualitative purposes, i.e. for determining the relative magnitude of the various terms in the equations. In an actual problem (cf. the similarity analysis), the reference quantities may depend upon the co-ordinates of the system. Thus, for the semi-infinite flat plate there is no constant reference length. However, on choosing $\tilde{x}$ instead of l for a reference length the non-dimensional longitudinal co-ordinate becomes of order $O(1)$. Since one may consider quite general reference temperatures and magnetic fields, it is clear that U may be a function of the independent variables. When determining the relative magnitude of various terms in the governing equations, constant reference quantities may be used. Thus in (3), l, U, etc., are assumed to be constants.

The non-dimensional equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$T - u + \frac{1}{H^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{G}{H^4}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}\right) = 0, \tag{5}$$

$$\frac{H^2}{G}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \frac{\partial p}{\partial y} = 0, \tag{6}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{PG}{H^2}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = 0. \tag{7}$$

These are the equations of continuity, x and y momentum and energy respectively. Compression work and viscous dissipation have been neglected in (7). Due to the process of non-dimensionalization three characteristic numbers of this problem occur in (4)–(7), viz.

the Hartmann number, $$H = (Al^2/\nu)^{\frac{1}{2}}, \tag{8}$$

the Grashof number, $$G = \frac{g\beta(T_W - T_\infty)\, l^3}{\nu^2}, \tag{9}$$

the Prandtl number, $$P = \nu/\kappa. \tag{10}$$

Here ν is the kinematic viscosity and κ the coefficient of thermal diffusion. In the course of the analysis specific conditions which have to be satisfied by these parameters in order that boundary-layer approximations be justified, will be derived. Special attention should be given to the first two terms of (5), which represent the buoyancy force and the magnetic force respectively. It is seen that these forces are acting in opposite directions and that they are of the same order of magnitude. The application of subsequent transformations will not change this qualitative equality, thus stressing the fact that we investigate free convection in the region where both the magnetic and the buoyancy force dominate the flow.

In the present paper we shall not solve the complete equations (4)–(7). Instead we shall try to derive suitable boundary-layer approximations. In the outer layer, which comprises the main part of the flow, the traditional boundary-layer condition $\partial^2/\partial x^2 \ll \partial^2/\partial y^2$ should hold. Thus, the solution of Singh & Cowling (1963) can be used as a first approximation. Since the existence of the inner layer depends on the absence of fluid slip at the boundary, the boundary-layer approximations should hold there *a fortiori*. It is then the object of this study to investigate higher approximations due to the interaction of the inner and the outer region. We shall not include in the solution higher approximations of the boundary layer itself, i.e. the influence of the inviscid entrainment flow, existing outside the outer region, will not be taken into account. Corrections upon the boundary layer are assumed to be an order of magnitude smaller than the perturbations presented. In any case, inclusion of this effect would necessitate presentation of a quadruple series expansion rather than the double series expansion given in this paper.

Outer layer

Singh & Cowling (1963) and Riley (1964) considering Prandtl numbers of order unity, stressed that the convection and the conduction terms of the energy equation are of the same order of magnitude in the dominant outer layer. The influence of the viscous stresses entered the solution through higher perturbations only. It has to be expected that this will be even more true for $P \ll 1$. In order to derive the boundary-layer equations for this region the following considerations have to be taken into account. Since x, u, T and p are of order one by virtue of (3), the order of these quantities has to remain unchanged through the boundary-layer transformation. As a result only v and y can be stretched. The stretching of these variables is restricted, however, by the condition that both terms in the equation of continuity (4) must remain of the same order of magnitude. Thus the quantities of the outer boundary layer are related to those of (4)–(7) by the following relations:

$$x_0 = x, \quad y_0 = \tau y, \quad u_0 = u, \quad v_0 = \tau v, \quad T_0 = T, \quad p_0 = p. \tag{11}$$

The subscript 0 refers to the outer layer. The value of τ is determined by the condition stated above: equality of order of conductive and convective heating in the outer layer. This gives

$$\tau = \left\{\frac{PG}{H^2}\right\}^{\frac{1}{2}}. \tag{12}$$

Through substitution of (11) and (12) into (4)–(7) the outer equations are obtained:

$$\frac{\partial u_0}{\partial x_0}+\frac{\partial v_0}{\partial y_0}=0, \tag{13}$$

$$T_0-u_0-\frac{G}{H^4}\left(u_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial u_0}{\partial y_0}+\frac{\partial p_0}{\partial x_0}\right)+\frac{PG}{H^4}\left(\frac{\partial^2 u_0}{\partial y_0^2}+\frac{H^2}{PG}\frac{\partial^2 u_0}{\partial x_0^2}\right)=0, \tag{14}$$

$$\frac{\partial p_0}{\partial y_0}=\frac{H^2}{PG}\left\{-u_0\frac{\partial v_0}{\partial x_0}-v_0\frac{\partial v_0}{\partial y_0}+P\left(\frac{\partial^2 v_0}{\partial y_0^2}+\frac{H^2}{PG}\frac{\partial^2 v_0}{\partial x_0^2}\right)\right\}, \tag{15}$$

$$\frac{\partial^2 T_0}{\partial y_0^2}-u_0\frac{\partial T_0}{\partial x_0}-v_0\frac{\partial T_0}{\partial y_0}+\frac{H^2}{PG}\frac{\partial^2 T_0}{\partial x_0^2}=0. \tag{16}$$

It is clear that here these equations become boundary-layer equations only if

$$\frac{H^2}{PG}\ll 1. \tag{17}$$

This is in contradiction to the analysis of Riley (1964), who used the boundary-layer approximations for values of a coefficient B much smaller than unity, where $B = PG/H^2$: in fact he used B as the singular perturbation expansion variable. Later the same procedure was partly followed by D'Sa. That Riley's assumption of small B cannot possibly hold can easily be seen from his definition of the boundary-layer similarity variable. Indeed, from his (5) we find for that variable

$$B^{\frac{1}{2}}y/x^{\frac{1}{4}}. \tag{18}$$

Now, for (18) to be a boundary-layer similarity variable with respect to the original unstretched co-ordinates x and y, B must be very *large*, which is in agreement with (17).

Thus, if condition (17) holds we easily find the boundary-layer equations which govern the flow in the outer region:

$$T_0-\frac{\partial\psi_0}{\partial y_0}-\frac{G}{H^4}\left(\frac{\partial\psi_0}{\partial y_0}\frac{\partial^2\psi_0}{\partial x_0\partial y_0}-\frac{\partial\psi_0}{\partial x_0}\frac{\partial^2\psi_0}{\partial y_0^2}\right)+\frac{PG}{H^4}\frac{\partial^3\psi_0}{\partial y_0^3}=0, \tag{19}$$

$$\frac{\partial^2 T_0}{\partial y_0^2}-\frac{\partial\psi_0}{\partial y_0}\frac{\partial T_0}{\partial x_0}+\frac{\partial\psi_0}{\partial x_0}\frac{\partial T_0}{\partial y_0}=0. \tag{20}$$

Here we have solved the equation of continuity by the introduction of the stream function ψ. Equation (15) shows that the pressure is constant through the boundary layer, but this constant is equal to zero on account of the outer boundary conditions. For $y_0\to\infty$ two boundary conditions are given, namely

$$T_0\to 0,\quad \frac{\partial\psi_0}{\partial y_0}\to 0\quad\text{as}\quad y_0\to\infty. \tag{21}$$

In addition, we have to find the behaviour of T_0 and ψ_0 as $y_0\to 0$, This behaviour can be determined only through matching with the solution that is valid in the inner boundary-layer region. The matching conditions will be formulated below.

From the fact that the no-slip condition at $y = 0$ is not fulfilled by ψ_0 we may immediately infer that the viscous term in (19) must be much smaller than the main terms of this equation. Since T_0, ψ_0 and all derivatives are of order $O(1)$, by definition this leads to the following condition:

$$PG/H^4 \ll 1. \tag{22}$$

This may be combined with (17) to yield bounds upon the analysis as follows,

$$1 \ll PG/H^2 \ll H^2, \tag{23}$$

or alternatively as

$$1 \ll H^2 \ll R \ll H^4. \tag{24}$$

$R = PG$ is the Rayleigh number. This is again in contradiction to the results of Riley and D'Sa. These authors applied boundary-layer theory while assuming Rayleigh numbers of order $O(1)$.

Inner layer

We have remarked earlier that Riley introduced an inner layer near the surface of the vertical flat plate in order to satisfy the no-slip condition, which is violated by the solution of Singh & Cowling. Naturally, in this layer the viscous term of the momentum equation should be of the same order of magnitude as the other main terms of that equation. Moreover, the buoyancy force and the magnetic force as well should be of the same order of magnitude in this layer. This can be understood through the following reasoning.

In the absence of viscous stresses only the outer layer will exist. Then the velocity at the wall will equal the characteristic velocity given in (1). If the fluid is slightly viscous there will be an inner layer within which the velocity increases asymptotically from zero to the characteristic velocity U imposed near the wall by the outer flow. So at the outer edge of the inner layer the magnetic force, which is proportional to U, reaches its maximum value. The buoyancy force reaches its maximum value at the wall. This shows that both forces are of the same order of magnitude in the inner layer. Riley has shown that this reasoning holds even for $P \sim 1$.

In order to derive the inner boundary-layer equations from the complete Navier–Stokes and energy equations (4)–(7) we will proceed in the same manner as for the outer boundary-layer equations. First, it follows from (3) that x, u, T and p are of order $O(1)$, by definition. So for the inner variables we have to obtain equations analogous to (11). The stretching constant is now determined by the condition that the viscous term be of the same order of magnitude as the magnetic and buoyancy terms. This gives

$$x_i = x, \quad y_i = Hy, \quad u_i = u, \quad v_i = Hv, \quad T_i = T, \quad p_i = p. \tag{25}$$

The inner equations are now easily found to be

$$\frac{\partial u_i}{\partial x_i} + \frac{\partial v_i}{\partial y_i} = 0, \tag{26}$$

$$T_i - u_i + \frac{\partial^2 u_i}{\partial y_i^2} - \frac{G}{H^4}\left(u_i \frac{\partial u_i}{\partial x_i} + v_i \frac{\partial u_i}{\partial y_i} + \frac{\partial p_i}{\partial x_i}\right) + \frac{1}{H^2}\frac{\partial^2 u_i}{\partial x_i^2} = 0, \tag{27}$$

$$\frac{\partial p_i}{\partial y_i} = \frac{H^2}{G}\left(\frac{\partial^2 v_i}{\partial y_i^2} + \frac{1}{H^2}\frac{\partial^2 v_i}{\partial x_i^2}\right) - \frac{1}{H^2}\left(u_i \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial v_i}{\partial y_i}\right), \tag{28}$$

$$\frac{\partial^2 T_i}{\partial y_i^2} - \frac{PG}{H^4}\left(u_i \frac{\partial T_i}{\partial x_i} + v_i \frac{\partial T_i}{\partial y_i}\right) + \frac{1}{H^2}\frac{\partial^2 T_i}{\partial x_i^2} = 0. \tag{29}$$

These equations reduce to boundary-layer equations provided

$$H^2 \gg 1. \tag{30}$$

Since $P \sim 1$ or $P < 1$ it follows from (17), (28), (30) that the pressure p_i does not vary across the inner boundary layer either. It must be noted that (30) is already contained in (24), so it does not restrict the range of the characteristic parameters any further.

The inner boundary-layer equations are now

$$T_i - \frac{\partial \psi_i}{\partial y_i} + \frac{\partial^3 \psi_i}{\partial y_i^3} - \frac{G}{H^4}\left(\frac{\partial \psi_i}{\partial y_i}\frac{\partial^2 \psi_i}{\partial x_i \partial y_i} - \frac{\partial \psi_i}{\partial x_i}\frac{\partial^2 \psi_i}{\partial y_i^2}\right) = 0, \tag{31}$$

$$\frac{\partial^2 T_i}{\partial y_i^2} - \frac{PG}{H^4}\left(\frac{\partial \psi_i}{\partial y_i}\frac{\partial T_i}{\partial x_i} - \frac{\partial \psi_i}{\partial x_i}\frac{\partial T_i}{\partial y_i}\right) = 0. \tag{32}$$

Boundary conditions can only be given explicitly at $y_i = 0$ namely,

$$T_i = 1, \quad \frac{\partial \psi_i}{\partial y_i} = 0, \quad \psi_i = 0 \quad \text{at} \quad y_i = 0. \tag{33}$$

For $y_i \to \infty$ additional boundary conditions have to be established through matching with the outer solution.

Matching

It is known that the ratio of the thicknesses of inner to outer layer determines the proper expansion variable for problems of the type considered here. As the thicknesses of the layers are determined by $y_i = O(1)$ and $y_0 = O(1)$ respectively one finds for the expansion variable of the singular perturbation problem (see (11) and (25)),

$$\epsilon_s = \{PG/H^4\}^{\frac{1}{2}}, \tag{34}$$

which measures the rate of interaction of both layers. There is, however, another small parameter in (19) and (31), namely

$$\epsilon_r = G/H^4, \tag{35}$$

which will give rise to a regular perturbation. Clearly the problem has to be solved by double expansion, which is partly singular and partly regular. The matching is established by the formulae

$$\psi_0(x_0, y_0, \epsilon_s, \epsilon_r) = \epsilon_s \psi_i(x_i, y_i, \epsilon_s, \epsilon_r), \tag{36}$$

$$T_0(x_0, y_0, \epsilon_s, \epsilon_r) = T_i(x_i, y_i, \epsilon_s, \epsilon_r), \tag{37}$$

$$x_0 = x_i = x, \quad y_0 = \epsilon_s y_i. \tag{38}$$

The outer problem is solved by

$$\psi_0 = \psi_0^{(0)}(x, y_0, \epsilon_r) + \ldots + \Delta_0^{(n)}(\epsilon_s)\,\psi_0^{(n)}(x, y_0, \epsilon_r) + \ldots. \tag{39}$$

For the temperature an analogous expression has to be introduced. The expansion parameters have to satisfy the conditions,

$$\operatorname{ord}(\Delta^{(n+1)}) < \operatorname{ord}(\Delta^{(n)}), \quad \Delta^{(0)} = 1. \tag{40}$$

The components of the expansion (39) may be regularly expanded in a series in the small parameter ϵ_r

$$\psi_0^{(n)}(x, y_0, \epsilon_r) = \sum_{m=0}^{\infty} \psi_0^{(n,m)}(x, y_0)\,\epsilon_r^m. \tag{41}$$

For the solution of the inner problem a similar set of expansions has to be given. Note that often the regular expansion (41) has to be modified through inclusion of logarithmic terms. See Stewartson (1957).

3. 'Power-law' wall temperatures and magnetic fields

Power-law boundary conditions have been explored widely in boundary-layer theory. The reason for this is that they often give rise to simple similarity transformations. In complicated problems such as the present one, power relationships lead to simple zeroth perturbations. Therefore, the following wall temperatures and magnetic fields will initially be considered

$$T_W = T_\infty + N\tilde{x}^k, \tag{42}$$

$$A = \alpha\tilde{x}^s. \tag{43}$$

Since the present configuration is devoid of a characteristic length, $\tilde{x}$ will assume the role of l. On replacing l by $\tilde{x}$ in (8) and (9) the local Hartmann number H_x and the local Grashof number G_x are obtained. Using a reference length which is thus variable has certain consequences for the analysis: we cannot set out directly from the inner and outer equations obtained from the formal analysis of § 2, since those had been derived by employing constant reference quantities. For the present we have to take up the original equations in dimensional form. On making the local Grashof and Hartmann numbers satisfy conditions analogous to those of (17) and (30), these will give rise to the following,

$$\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\,\partial\tilde{y}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} = \nu\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3} + g\beta(\tilde{T} - T_\infty) - A\frac{\partial\tilde{\psi}}{\partial\tilde{y}}, \tag{5a}$$

$$\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{T}}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{T}}{\partial\tilde{y}} = \kappa\frac{\partial^2\tilde{T}}{\partial\tilde{y}^2}. \tag{7a}$$

We can now use the information of (11) and (12) to define the dimensionless outer functions F and ϑ,

$$\tilde{\psi} = \nu\left\{\frac{G_x}{PH_x^2}\right\}^{\frac{1}{2}} F(\mu, \epsilon_s, \epsilon_r), \tag{44}$$

$$\tilde{T} = T_\infty + N\tilde{x}^k\,\vartheta(\mu, \epsilon_s, \epsilon_r), \tag{45}$$

where μ, ϵ_s and ϵ_r are given by

$$\mu = \frac{\tilde{y}}{\tilde{x}}\left\{\frac{PG_x}{H_x^2}\right\}^{\frac{1}{2}}, \tag{46}$$

$$\epsilon_s = \left\{\frac{PG_x}{H_x^4}\right\}^{\frac{1}{2}}, \tag{47}$$

$$\epsilon_r = \frac{G_x}{H_x^4}. \tag{48}$$

Upon substituting (44)–(48) in (5*a*) and (7*a*) we obtain

$$\vartheta - \frac{\partial F}{\partial \mu} = \epsilon_r\left\{(k-s)\left(\frac{\partial F}{\partial \mu}\right)^2 - \frac{k-s+1}{2}F\frac{\partial^2 F}{\partial \mu^2}\right\} + \epsilon_r^2(k-2s-1)\left\{\frac{\partial F}{\partial \mu}\frac{\partial^2 F}{\partial \mu\,\partial \epsilon_r} - \frac{\partial F}{\partial \epsilon_r}\frac{\partial^2 F}{\partial \mu^2}\right\}$$

$$+\tfrac{1}{2}\epsilon_r\epsilon_s(k-2s-1)\left\{\frac{\partial F}{\partial \mu}\frac{\partial^2 F}{\partial \mu\,\partial \epsilon_s} - \frac{\partial F}{\partial \epsilon_s}\frac{\partial^2 F}{\partial \mu^2}\right\} - \epsilon_s^2\frac{\partial^3 F}{\partial \mu^3}, \qquad (49)$$

$$\frac{\partial^2 \vartheta}{\partial \mu^2} + \frac{k-s+1}{2}F\frac{\partial \vartheta}{\partial \mu} - k\vartheta\frac{\partial F}{\partial \mu} = \epsilon_r(k-2s-1)\left\{\frac{\partial F}{\partial \mu}\frac{\partial \vartheta}{\partial \epsilon_r} - \frac{\partial F}{\partial \epsilon_r}\frac{\partial \vartheta}{\partial \mu}\right\}$$

$$+\tfrac{1}{2}\epsilon_s(k-2s-1)\left\{\frac{\partial F}{\partial \mu}\frac{\partial \vartheta}{\partial \epsilon_s} - \frac{\partial F}{\partial \epsilon_s}\frac{\partial \vartheta}{\partial \mu}\right\}. \qquad (50)$$

The transformation to inner variables can be obtained directly through (36)–(38)

$$F(\mu, \epsilon_s, \epsilon_r) = \epsilon_s f(\eta, \epsilon_s, \epsilon_r), \qquad (51)$$

$$\vartheta(\mu, \epsilon_s, \epsilon_r) = \theta(\eta, \epsilon_s, \epsilon_r), \qquad (52)$$

$$\mu = \epsilon_s \eta. \qquad (53)$$

Through the introduction of (51), (53) into (49), (50) the inner equations are obtained

$$\frac{\partial^3 f}{\partial \eta^3} - \frac{\partial f}{\partial \eta} + \theta = \epsilon_r\left\{(k-s)\left(\frac{\partial f}{\partial \eta}\right)^2 + (\tfrac{3}{2}s-k)f\frac{\partial^2 f}{\partial \eta^2}\right\}$$

$$+\epsilon_r^2(k-2s-1)\left\{\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta\,\partial \epsilon_r} - \frac{\partial f}{\partial \epsilon_r}\frac{\partial^2 f}{\partial \eta^2}\right\} + \tfrac{1}{2}\epsilon_r\epsilon_s(k-2s-1)\left\{\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta\,\partial \epsilon_s} - \frac{\partial f}{\partial \epsilon_s}\frac{\partial^2 f}{\partial \eta^2}\right\}, \qquad (54)$$

$$\frac{\partial^2 \theta}{\partial \eta^2} = \epsilon_s^2\left\{k\theta\frac{\partial f}{\partial \eta} + (\tfrac{3}{2}s-k)f\frac{\partial \theta}{\partial \eta}\right\} + \epsilon_r\epsilon_s^2(k-2s-1)\left\{\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \epsilon_r} - \frac{\partial f}{\partial \epsilon_r}\frac{\partial \theta}{\partial \eta}\right\}$$

$$+\tfrac{1}{2}\epsilon_s^3(k-2s-1)\left\{\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \epsilon_s} - \frac{\partial f}{\partial \epsilon_s}\frac{\partial \theta}{\partial \eta}\right\}. \qquad (55)$$

If the left-hand sides of (51) and (52) are evaluated for small values of μ and the right-hand sides for large values of η matching can be established by using (53).

Special cases

Power law wall temperatures and magnetic fields include a number of important special cases. Some of these have been considered previously but, for reasons given in the first part of this paper, should be reconsidered.

(i) $k = s = 0$: ***both wall temperature and magnetic field uniform.*** This problem was considered in detail by Riley (1964). As we have remarked earlier the expansion variables given here differ fundamentally from those used by Riley. In fact, it has been proved that that author's expansion variable B must be large to justify application of the boundary-layer approximations. Therefore we will not be able to compare the results of the present analysis with those of Riley. However, the method of solution employed by him has many elements in common with the present method. We, therefore, need not elaborate on the algebraical details. In effect some of the perturbation equations coincide with the corresponding ones of Riley. One feature of his work was the occurrence of a logarithmic expansion term, the reason for which has been amply discussed by him.

Analogously we may prove that the present outer expansion should be given as

$$F = F^{(0,0)}(\mu) + \epsilon_s F^{(1,0)}(\mu) + \epsilon_r \ln \epsilon_r F_l^{(0,1)}(\mu) + \epsilon_r F^{(0,1)}(\mu) + \ldots, \tag{56}$$

$$\vartheta = \vartheta^{(0,0)}(\mu) + \epsilon_s \vartheta^{(1,0)}(\mu) + \epsilon_r \ln \epsilon_r \vartheta_l^{(0,1)}(\mu) + \epsilon_r \vartheta^{(0,1)}(\mu) + \ldots, \tag{57}$$

while the inner expansion starts with

$$f = f^{(0,0)}(\eta) + \epsilon_s f^{(1,0)}(\eta) + \ldots, \tag{58}$$

$$\theta = \theta^{(0,0)}(\eta) + \epsilon_s \theta^{(1,0)}(\eta) + \ldots, \tag{59}$$

where, for the present, we have left out of the inner expansion perturbations with respect to ϵ_r. Upon substitution of the expansions (56)–(59) in the appropriate outer and inner equations (49)–(50) and (54)–(55) a set of ordinary differential equations will be obtained for the coefficients of these expansions. Determining the fundamental term and the terms proportional to ϵ_s is fairly straightforward if the matching rule is applied properly. We find, as did Riley,

$$f^{(0,0)} = \eta - 1 + e^{-\eta}, \quad \theta^{(0,0)} = 1, \quad f^{(1,0)} = (\gamma/2)\eta^2, \quad \theta^{(1,0)} = \gamma\eta. \tag{60}$$

Here $\gamma = F^{(0,0)''}(0) = -0{\cdot}4437$ and primes denote differentiation with respect to the argument. $F^{(0,0)}(\mu)$ satisfies Blasius's equation

$$F^{(0,0)'''} + \tfrac{1}{2} F^{(0,0)} F^{(0,0)''} = 0, \tag{61}$$

with boundary conditions for a moving surface in an infinite expanse of fluid at rest. Furthermore, we have

$$F^{(1,0)}(\mu) = -1, \quad \vartheta^{(1,0)}(\mu) = 0. \tag{62}$$

Riley introduced an expansion term similar to $\epsilon_r \ln \epsilon_r$ in order to ensure the existence of solutions that decay exponentially as $\mu \to \infty$. Our $F_l^{(0,1)}$ and $\vartheta_l^{(0,1)}$ satisify the same equations as do the analogous functions of Riley. The solution that behaves exponentially for $\mu \to \infty$ and which satisfies the obvious matching condition $F_l^{(0,1)}(0) = 0$ is

$$F_l^{(0,1)} = \xi(F^{(0,0)} - \mu F^{(0,0)'}), \tag{63}$$

$$\vartheta_l^{(0,1)} = -\xi \mu F^{(0,0)''}. \tag{64}$$

ξ is a constant which is determined by requiring that $\vartheta^{(0,1)}$ and $F^{(0,1)}$ tend exponentially to their respective asymptotic values as $\mu \to \infty$. The equations for $\vartheta^{(0,1)}$ and $F^{(0,1)}$ are

$$\vartheta^{(0,1)} - F^{(0,1)'} = -\tfrac{1}{2} F^{(0,0)} F^{(0,0)''}, \tag{65}$$

$$\vartheta^{(0,1)''} + \tfrac{1}{2} F^{(0,0)} \vartheta^{(0,1)'} + F^{(0,0)'} \vartheta^{(0,1)} - \tfrac{1}{2} \vartheta^{(0,0)'} F^{(0,1)} = -F^{(0,0)'} \vartheta_l^{(0,1)} + F_l^{(0,1)} \vartheta^{(0,0)'}. \tag{66}$$

Combination of (65) and (66) and making use of (61), (63) and (64) leads to a differential equation for $F^{(0,1)}$ alone

$$F^{(0,1)'''} + \tfrac{1}{2} F^{(0,0)} F^{(0,1)''} + F^{(0,0)'} F^{(0,1)'} - \tfrac{1}{2} F^{(0,0)''} F^{(0,1)} = \tfrac{1}{2}\{F^{(0,0)''}\}^2 - 2\xi F^{(0,0)'''}. \tag{67}$$

Referring to Riley's work for the proof, we know that the following integral relation must hold if $F^{(0,1)}$ is to have an exponential behaviour for $\mu \to \infty$

$$\int_0^\infty F^{(0,0)}[\tfrac{1}{2}\{F^{(0,0)''}\}^2 - 2\xi F^{(0,0)'''}]\,d\mu = 0. \tag{68}$$

Hence we find

$$\xi = \tfrac{1}{2}\gamma^2, \tag{69}$$

since $F^{(0,0)'}(0) = 1$. This completes the solution of (63)–(64).

Riley following Stewartson (1957) has shown that the solution to (67) cannot be determined uniquely, since eigensolutions such as (63) and (64) can always be added to a particular solution. Thus, it is not very useful to consider $F^{(0,1)}$ and $\vartheta^{(0,1)}$ in any further detail.

The outer expansions can now finally be given as

$$F = F^{(0,0)}(\mu) - \epsilon_s + \tfrac{1}{2}\gamma^2 \epsilon_r \ln \epsilon_r (F^{(0,0)}(\mu) - \mu F^{(0,0)'}(\mu)) + \ldots, \tag{70}$$

$$\vartheta = \vartheta^{(0,0)}(\mu) - \tfrac{1}{2}\gamma^2 \epsilon_r \ln \epsilon_r \mu F^{(0,0)''}(\mu) + \ldots, \tag{71}$$

The question about the extension of the inner expansions (58) and (59) to perturbations with respect to ϵ_r has still to receive attention. It is reasonable to assume that the terms,

$$f_l^{(0,1)}(\eta)\,\epsilon_r \ln \epsilon_r + f^{(0,1)}(\eta)\,\epsilon_r, \tag{72}$$

should be added to complete the expansions to and including first-order corrections. Upon substituting the inner expansions into the governing (54) and (55) it is seen that the functions of (72) that satisfy the boundary conditions at the wall (perturbations of f, f' and θ zero at $\eta = 0$) are

$$f_l^{(0,1)} = \tfrac{1}{2}A\eta^2, \quad f^{(0,1)} = \tfrac{1}{2}C\eta^2, \quad \theta_l^{(0,1)} = A\eta, \quad \theta^{(0,1)} = C\eta, \tag{73}$$

where A and C are constants that have to be determined through matching. As an illustrative example let us take matching with $\theta^{(0,1)}$. For $\eta \to \infty$ this term gives rise to a factor $C\eta\epsilon_r$ on the right-hand side of (52). Upon writing this term as $C\mu(\epsilon_r/\epsilon_s)$ it follows that it can never be furnished by the left-hand side of (52) for $\mu \to \infty$ if $C \neq 0$; thus we have $C = 0$. In the same manner, one can prove $A = 0$. Therefore the terms of (72) should not be included in the inner expansion.

(ii) $k = 1$, $m = 0$: *wall-temperature varying linearly with* $\tilde{x}$ *and a uniform magnetic field*. This problem was investigated in detail for $P \sim 1$ by D'Sa (1967). In the early stages of his analysis D'Sa follows Riley closely, considering boundary-layer equations, but assuming the Rayleigh number of order $O(1)$ and defining an expansion variable $R/H^2 \ll 1$. However, when presenting the case $k = 1$ and $m = 0$ this author changes his expansion variable into $R^{\frac{1}{2}}/H^2 = \epsilon_s$ which we have found in the preceding to be the correct choice. As D'Sa only considered Prandtl numbers of order $O(1)$ there was no need for him to introduce a second expansion variable. The Prandtl number naturally appeared as a coefficient in his solution.

We will present the result of our own analysis without the detailed calculations through which it was obtained. The special interest of this case is, that it admits of a complete analytical solution. Moreover, this solution does not require the inclusion of irregular, e.g. logarithmic, expansion variables. Only integer powers of ϵ_r and ϵ_s are encountered. Therefore there are also no constants which remain indeterminate, as was the case with $k = s = 0$. Up to and including the orders $O(\epsilon_s^2)$, $O(\epsilon_r \epsilon_s)$ and $O(\epsilon_r^2)$ we find

$$F^{(0,0)} = 1 - e^{-\mu}; \quad \vartheta^{(0,0)} = e^{-\mu},$$

$$F^{(1,0)} = -\tfrac{1}{2} - \tfrac{1}{2}(1+\mu)\, e^{-\mu}; \quad \vartheta^{(1,0)} = \tfrac{1}{2}\mu e^{-\mu},$$

$$F^{(0,1)} = -\tfrac{1}{2} + \tfrac{1}{2}(1-\mu)\, e^{-\mu}; \quad \vartheta^{(0,1)} = \tfrac{1}{2}\mu e^{-\mu},$$

$$F^{(2,0)} = \frac{9}{8} + \frac{-9+7\mu-\mu^2}{8} e^{-\mu}; \quad \vartheta^{(2,0)} = \frac{\mu^2-9\mu+8}{8} e^{-\mu},$$

$$F^{(1,1)} = \frac{13}{8} + \frac{5+13\mu-2\mu^2}{8} e^{-\mu}; \quad \vartheta^{(1,1)} = \frac{2\mu^2-13\mu}{8} e^{-\mu},$$

$$F^{(0,2)} = \frac{7}{8} + \frac{-7+9\mu-\mu^2}{8} e^{-\mu}; \quad \vartheta^{(0,2)} = \frac{\mu^2-7\mu}{8} e^{-\mu},$$

$$f^{(0,0)} = \eta - 1 + e^{-\eta}, \quad \theta^{(0,0)} = 1,$$

$$f^{(1,0)} = -\tfrac{1}{2}\eta^2, \quad \theta^{(1,0)} = -\eta,$$

$$f^{(0,1)} = \tfrac{9}{4} - \eta - (\tfrac{1}{4}\eta^2 + \tfrac{5}{4}\eta + \tfrac{9}{4})\, e^{-\eta}, \quad \theta^{(0,1)} = 0,$$

$$f^{(2,0)} = -\tfrac{5}{2} + 2\eta + \tfrac{1}{4}\eta^2 + \tfrac{1}{6}\eta^3 + \tfrac{1}{2}(\eta+5)\, e^{-\eta}, \quad \theta^{(2,0)} = 1 + \tfrac{1}{2}\eta + \tfrac{1}{2}\eta^2 - e^{-\eta},$$

$$f^{(1,1)} = -\tfrac{31}{8} + \eta + \tfrac{3}{4}\eta^2 + (\tfrac{31}{8} + \tfrac{23}{8}\eta + \tfrac{7}{8}\eta^2 + \tfrac{1}{12}\eta^3)\, e^{-\eta}, \quad \theta^{(1,1)} = \tfrac{1}{2}\eta,$$

$$f^{(0,2)} = -\tfrac{671}{96} + 2\eta - \tfrac{1}{12}e^{-2\eta} + (\tfrac{679}{96} + \tfrac{157}{32}\eta + \tfrac{53}{32}\eta^2 + \tfrac{13}{48}\eta^3 + \tfrac{1}{32}\eta^4)\, e^{-\eta}, \quad \theta^{(0,2)} = 0.$$

It can be shown, by expressing $\epsilon_r = \epsilon_s^2/P$ and by expanding both D'Sa's result and ours up to the order $O(\epsilon_s^2)$ with $P \sim 1$, that both analyses yield exactly the same result. Since D'Sa worked out his solution considering suction at the wall, his general result should be evaluated for the impermeable wall in this comparison.

(iii) *Heat source at leading edge of thermally insulated plate, arbitrary magnetic field.*† As we have seen in the previous examples, the main term of the inner expansion for the temperature is merely a constant which is usually normalized to unity. Therefore it does not contribute to the heat flux. As a consequence the main term of the outer expansion yields the major part of the heat flux even at the wall, since the second term of the inner expansion is equal to $\epsilon_s \eta \vartheta^{(0,0)\prime}(0)$. Thus for an adiabatic wall we have the condition $\vartheta^{(0,0)\prime}(0) = 0$. It is true that for the higher perturbations we have to add the conditions $\theta^{(n,m)\prime}(0) = 0$ for the insulated wall because in general these perturbations will no longer be linear in η.

Most important about a heat source problem is the wall-temperature distribution it gives rise to. It is precisely this point that can be clarified by considering the main term of the outer expansion. Through integration of (50), neglecting the terms proportional to ϵ_r and ϵ_s we obtain

$$-\vartheta^{(0,0)\prime}(0) + \frac{s-1-3k}{2} \int_0^\infty \vartheta^{(0,0)} F^{(0,0)\prime}\, d\mu = 0. \tag{74}$$

† This section will also describe the solution for a thin plate with flow symmetrical on both its sides.

Now, the integral in (74) can never vanish since it is related to the heat flux which is dissipated by the source of strength Q,

$$Q = \rho c_p \int_0^\infty \tilde{u}(\tilde{T} - T_\infty)\, d\tilde{y}. \tag{75}$$

ρ is the density and c_p the specific heat at constant pressure. Therefore, $\vartheta^{(0,0)\prime}(0)$ can be zero only if

$$k = (s-1)/3. \tag{76}$$

By substituting (44)–(46) into (75) one may solve for the hitherto unknown constant N by taking only the main term of the outer expansion. It is reasonable to take the main term only, since the heat flux Q will remain the same even if ϵ_s and ϵ_r tend to zero. By substituting the value just found for N into (45), where one must be aware of the fact that N also occurs in G_x, we find for vanishing ϵ_r and ϵ_s

$$T_W = T_\infty + \left\{\frac{\alpha Q^2 \tilde{x}^{s-1}}{g\beta\rho c_p \lambda}\right\}^{\frac{1}{3}} \vartheta^{(0,0)}(0) + \text{higher orders}. \tag{77}$$

For convenience we have here replaced the normalization $\vartheta^{(0,0)}(0) = 1$ by

$$\int_0^\infty \vartheta^{(0,0)} F^{(0,0)\prime}\, d\mu = 1. \tag{78}$$

From (49) and (50) it follows immediately that the first-order outer equations are

$$\vartheta^{(0,0)} - F^{(0,0)\prime} = 0, \tag{79}$$

$$\vartheta^{(0,0)\prime\prime} + \tfrac{1}{3}(1-s)F^{(0,0)}\vartheta^{(0,0)\prime} + \tfrac{1}{3}(1-s)F^{(0,0)\prime}\vartheta^{(0,0)} = 0. \tag{80}$$

One can solve these equations analytically using (78),

$$F^{(0,0)}(0) = 0 \quad \text{and} \quad \vartheta^{(0,0)}(\infty) = 0,$$

giving

$$F^{(0,0)}(\mu) = \left(\frac{9}{1-s}\right)^{\frac{1}{3}} \tanh\left\{\mu \frac{(1-s)^{\frac{2}{3}}}{2 . 3^{\frac{1}{3}}}\right\}, \tag{81}$$

$$\vartheta^{(0,0)}(\mu) = \left\{\frac{3(1-s)}{8}\right\}^{\frac{1}{3}} \cosh^{-2}\left\{\frac{\mu(1-s)^{\frac{2}{3}}}{2 . 3^{\frac{1}{3}}}\right\}, \tag{82}$$

from which it is easy to derive the unknown constant of (77).

The search for higher approximations is developed along exactly the same lines as were given during the discussion of the previous problems. This does not yield any information that is fundamentally new.

There is some point in discussing the admissible values of s for this example. It is clear from the solution that s should not exceed the value 1. Later the range of admissible values of k and s will be investigated thoroughly. Anticipating the results of this we may give the range of values of s for which the present example is meaningful.

$$-\tfrac{4}{5} \leqslant s < 1. \tag{83}$$

It is known from the literature (Sparrow & Gregg 1958, Kuiken 1967) that in the absence of a magnetic field the wall plume (heat source at the leading edge of an insulated vertical plate) gives rise to a wall-temperature distribution that behaves like $\tilde{x}^{-\frac{3}{5}}$. It is seen that for all values of s given by (83) the temperature decay in the direction $\tilde{x} \to \infty$ will be slower than, or at most equal to, that obtained in the absence of a magnetic field. This has to be expected indeed as the field will slow down the flow and will thus reduce the convective cooling.

(iv) *Uniform heat flux: arbitrary magnetic field.* Other important examples emerge through an extension of the boundary condition (42) and (43). Due to the nature of the solution, which are series expansions, it is possible to consider the more general boundary conditions,

$$T_W = T_\infty + N\tilde{x}^k h(\tilde{x}), \tag{84}$$

$$A = \alpha\tilde{x}^s l(\tilde{x}), \tag{85}$$

where $h(\tilde{x})$ and $l(\tilde{x})$ can be expanded in exactly the same way as the solutions of (39) and (41). Instead of the homogeneous boundary conditions that have to be satisfied by the higher perturbations in case the wall temperatures of (42) are used, we now encounter non-homogeneous ones according to the expansion of $h(\tilde{x})$. Examples of a similar procedure are given in a recent paper (Kuiken 1968). As the present problem involves singular perturbations, the exact form of the series solution (39) is not known beforehand. This may complicate matters somewhat, since $h(\tilde{x})$ must fit the expansion (39): it may limit the generality of choice of $h(\tilde{x})$.

As an important example, we may note the problem of uniform heat flux through the bounding wall, which often occurs in practical cases. Under such conditions the wall-temperature (84) is unknown initially, and instead

$$q = -\lambda \left.\frac{\partial \tilde{T}}{\partial \tilde{y}}\right|_{y=0} \tag{86}$$

is a given (uniform) heat flux through the surface of the plate. Here λ is the coefficient of thermal conduction. Just as in the previous example we shall confine our attention to the main term of the expansion, as this obviously is the most interesting feature of this problem. Again it suffices to consider the main term of the outer expansion, since the main term of the inner expansion merely gives a uniform temperature distribution. From (45), (46) and (86), neglecting higher approximations, we obtain

$$q \sim -\lambda N\tilde{x}^{k-1}\left\{\frac{PG_x}{H_x^2}\right\}^{\frac{1}{2}} \vartheta^{(0,0)\prime}(0). \tag{87}$$

Using the definitions of G_x and H_x it follows that q is independent of $\tilde{x}$ only if

$$k = (s+1)/3. \tag{88}$$

Here it is found most convenient to normalize the solution in such a way that $\vartheta^{(0,0)\prime}(0) = -1$. We can solve for the unknown parameter N using (87). By substituting finally in (84) we find for the approximate value of the wall temperature,

$$T_W = T_\infty + \left\{\frac{q^2\kappa\alpha\tilde{x}^{s+1}}{\lambda^2 g\beta}\right\}^{\frac{1}{3}} \vartheta^{(0,0)}(0) + \text{higher orders}. \tag{89}$$

The value of $\vartheta^{(0,0)}(0)$ can be obtained through numerical integration of the pertinent outer differential equations. For $s = \frac{1}{2}$ these differential equations admit of an analytical solution:

$$\vartheta^{(0,0)}(\mu) = 2^{\frac{1}{3}} \exp\{-\mu/2^{\frac{1}{3}}\}, \quad F^{(0,0)}(\mu) = 2^{\frac{2}{3}}(1-\exp\{-\mu/2^{\frac{1}{3}}\}), \tag{90}$$

from which it is easy to derive the unknown constant of (89).

4. Concluding remarks

In the course of the analysis several conditions have been derived that must be satisfied by the Hartmann, Grashof and Prandtl numbers and by combinations of these numbers. We may specify (17), (22), (23), (24) and (30). All these conditions are combined in the one condition (24) that was also derived by Singh & Cowling (1963). In addition to these bounds on H, G and P, which were found during the derivation of the boundary-layer approximations, we discuss here other restricting conditions through the requirement that the expansion variables ϵ_r and ϵ_s be small with respect to unity. Adopting the convention that $a \underbrace{< \; < \; \ldots \; <}_{n \text{ times}} b$ stand for $a \approx 10^{-n}b$ we could express this as $\epsilon_r < 1$ and $\epsilon_s < 1$. From (34)–(35) we then find

$$PG/H^4 \ll 1 \tag{91}$$

and

$$G/H^4 < 1. \tag{92}$$

According to (24) the inequality (91) does not restrict further the range of admissible values of H, G and P. So finally we have to contend with condition (24) and the relatively weak condition (92).

It is quite possible to give a practical example that satisfies the above conditions. Consider free convection of liquid mercury (approximate property values: $P \sim 0{\cdot}02$, $\beta \sim 10^{-4}/°\text{K}$, $\nu \sim 10^{-7}\ \text{m}^2/\text{s}$, $\rho \sim 10^4\ \text{kg/m}^3$) along a vertical plate of length $l \sim 0{\cdot}1$ m. Let a typical temperature difference be 50 °K; the induction B of the cross-field is $0{\cdot}1$ weber/m². With an electrical conductivity of $\sigma \sim 10^6$ mho/m the value of A is

$$A = \sigma B^2/\rho \sim 1. \tag{93}$$

The Hartmann, Grashof and Rayleigh numbers can now be obtained

$$H \sim 10^{\frac{5}{2}}, \quad G \sim 10^{10}/2, \quad R \sim 10^8. \tag{94}$$

It can easily be verified that these values satisfy the conditions (24) and (92). For the approximate values of the expansion parameters we find in this case $\epsilon_s \sim 0{\cdot}1$, $\epsilon_r \sim 0{\cdot}5$. Consequently, in spite of the fact that the Prandtl number is very low the first perturbations of the regular expansion are predominant over the singular ones.

It is of interest to note a fundamental difference between the two expansion parameters ϵ_r and ϵ_s. By expressing these quantities in the physical parameters of the system, through (8)–(10), we obtain

$$\epsilon_s = \left\{\frac{g\beta(T_W - T_\infty)}{\kappa A^2 l}\right\}^{\frac{1}{2}} \nu^{\frac{1}{2}}, \tag{95}$$

and

$$\epsilon_r = \frac{g\beta(T_W - T_\infty)}{A^2 l}. \tag{96}$$

It is seen that ϵ_s tends to zero with ν. ϵ_r, however, is not affected by this limiting process. Since the no-slip condition holds in the present analysis, the limit $\nu \rightarrow 0$ is obviously singular. This emphasizes the singular nature of ϵ_s, the expansion variable for the singular perturbation. ϵ_r gives rise to a regular perturbation.

It is necessary to analyze condition (24) for 'power-law' wall temperatures and magnetic fields. Both H and G involve powers of x. One has

$$H_x \sim x^{\frac{1}{2}(s+2)}, \quad G_x \sim x^{k+3}. \tag{97}$$

From (24) and (97) the necessary conditions for the present analysis to be valid far downstream are

$$k-s+1 > 0, \tag{98}$$

$$-k+2s+1 > 0, \tag{99}$$

and

$$s+2 > 0. \tag{100}$$

These inequalities are actually tantamount to $H_x^2/R_x \rightarrow 0$, $R_x/H_x^4 \rightarrow 0$ and $H_x^{-2} \rightarrow 0$ as $\tilde{x} \rightarrow \infty$. If instead of the inequality sign we use an equality sign in (99), the expansion variables both become parameters independent of x. The expressions (98) and (100) become identical in that case. Thus if $s > -2$ the solutions will be valid for all values of x for which the boundary-layer approximations are applicable. The problem for which $k = 1$ and $s = 0$, that we solved in § 3, belongs to this class.

It is easy to establish the conditions for solutions to be valid near the leading edge: It is required that k be always less than -3. Such cases have to be excluded from consideration, however, since on physical grounds one may state that k must be larger than -1. Thus the title problem will either admit of solutions which are valid downstream only, or it will yield results that can be used for all values of x. In figure 1 the conditions (98)–(100) are represented in a k–s diagram together with the condition $k > -1$. The range (I) of admissible values of k and s is clearly shown in this way.

As this analysis excluded higher approximations of the boundary layer from consideration, it is necessary to investigate the order of magnitude of the discarded terms in comparison with the small perturbations that were actually retained. A thorough investigation into this matter would require presentation of the boundary-layer expansions. This is outside the scope of the work presented here. We may obtain some insight into this matter by guessing the form of the boundary-layer expansion from the experience of other authors. It is known that the interaction of the boundary layer with the inviscid entrainment flow is the cause of existence of higher approximations to the basic boundary layer. Now in most cases this interaction is proportional to the relative thickness of the boundary layer. For pure free convection past a vertical plate we have this behaviour (Yang & Jerger 1964). As the inviscid entrainment is mainly caused by the outer boundary layer this relative thickness is determined by $y_0 = O(1)$. Thus the boundary-layer perturbations are proportional to

$$(H^2/PG)^{\frac{1}{2}}. \tag{101}$$

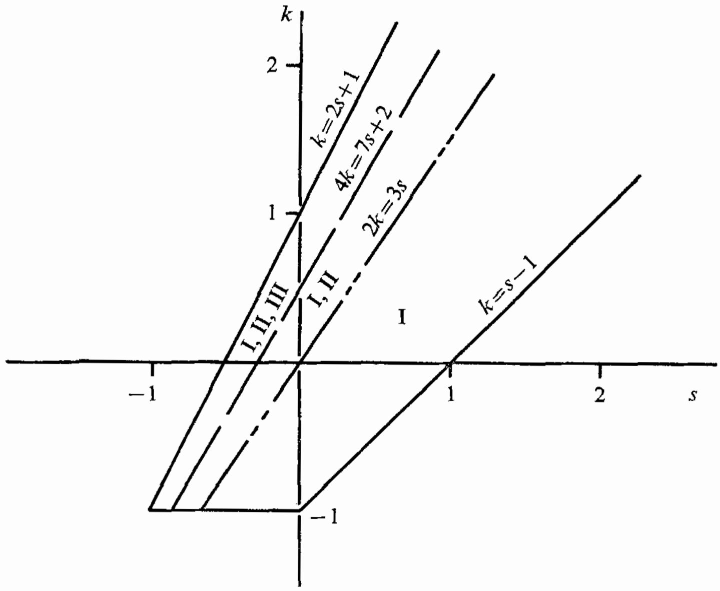

FIGURE 1. Region of validity of 'power-law' conditions.

Discarding this term with respect to ϵ_s requires

$$(H^2/PG)^{\frac{1}{2}} < (PG/H^4)^{\frac{1}{2}} \tag{102}$$

or

$$H^3 < R, \tag{103}$$

which may be interpreted as, say, $H^3 \sim R \times 10^{-1}$. For 'power-law' wall temperatures the effect of condition (103) has been given in figure 1, (II). As a further example it is easy to verify that neglecting (101) with respect to ϵ_s^3 would require $H^{\frac{7}{2}} \ll R$ (figure 1, (III)). The trend is clear: by extending the present expansion further and further without including higher order boundary-layer expansions, the range of admissible values of H and R, and therefore of k and s, becomes narrower. Carrying through the expansion for ϵ_r will lead to similar restrictions. Summing up, we may state that the conditions of (24) and (92) hold if only the main terms are retained. By adding further terms in the ϵ_r, ϵ_s expansions the conditions of applicability become more restrictive. Their upper bound is given by the right-hand side of (24).

The author thanks Dr Z. Rotem for discussions and help. He is indebted to the National Research Council of Canada and to Dr Rotem for financial support.

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